CS3621 Midterm Solution (Fall 2005)
150 points

1. **Geometric Transformation**

(a) [**5 points**] Find the 2D transformation matrix for the "reflection about the y-axis" transformation (*i.e.*, x goes to $-x$ and y goes to y). Is it an Euclidean transformation? Why?

Solution: Since $x' = -x$ and $y' = y$, the transformation matrix is

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

This is certainly an Euclidean transformation because Euclidean transformation include reflections, translations and rotations. ■

(b) [**10 points**] Find a projective transformation and its matrix that can map the points $(1,0)$ and $(-1,0)$, or $(1,0,1)$ and $(-1,0,1)$ in homogeneous coordinates, to infinity. What is the transformed result of the circle $x^2+y^2=1$ under your transformation? **Note that there are two questions in this problem.**

Solution: The key is to find a projective transformation that can map both $(1,0,1)$ and $(-1,0,1)$ to infinity (*i.e.*, the third component of the result being zero). Since both points have a 0 y-coordinatele, the third row of the desired projective transformation may have 0, 1 and 0. Thus, the projective transformation can have the following form:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 1 & 0 \end{bmatrix}$$

We can fill in the remaining six entries, as long as the determinant of the matrix is non-zero. An easy choice would be

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and we have

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, $(1,0,1)$ and $(-1,0,1)$ are mapped to $(1,1,0)$ $(-1,1,0)$, both being points at infinity. From the transformation matrix, we have

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

and, hence, $x' = x$, $y' = w$ and $w' = y$. Plugging these equations into the equation of the unit circle in homogenenous coordinate $x^2 + y^2 - w^2 = 0$ yields $(x')^2 + (w')^2 - (y')^2 = 0$. Rewriting this equation yields $-x^2 + y^2 = 1$. Thus, under the computed projective transformation, the unit circle is mapped to a hyperbola! ■

2. **Representations**

   (a) [**10 points**] If two small boxes are subtracted from a large box, list the numbers of vertices, edges, faces, loops, shells and genus. In POV-Ray's language, the solid is constructed as follows:

```
#declare  Object =
                difference {
                     box { < -1, -1, -1 >, < 1, 1, 1 > }
                     box { < -0.75, -0.75, -0.75 >, < -0.5, -0.5, -0.5 > }
                     box { < 0.5, 0.5, 0.5 >, < 0.75, 0.75, 0.75 > }
                }
```

   Use your finding to verify the Euler-Poincaré formula.

   Solution: The following table summarizes all required values:

Variable	*Value*	*Reason*
V	$8 \times 3 = 24$	there are three cubes each of which has 8 vertices
E	$12 \times 3 = 36$	there are three cubes each of which has 12 edges
F	$6 \times 3 = 18$	there are three cubes each of which has 6 faces
G	0	no penetrating holes
S	3	one outside and two inside
L	$18 = F$	no holes on each face

   Therefore, we have the following verification:

$$V - E + F - (L - F) - 2(S - G) = 24 - 36 + 18 - (18 - 18) - 2(3 - 0) = 0$$

   and the Euler-Poincaré holds. ∎

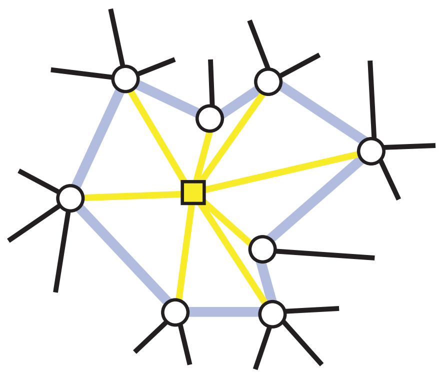
   (b) [**10 points**] Why are the *regularized* Boolean operators used for constructive solid geometry rather than the traditional set operators. Use examples to illustrate your point. **You must answer the question first, followed by examples to illustrate your answer. Otherwise, you will receive no credit.**

   Solution: The result of applying the conventional set Boolean operators (*i.e.*, set union, intersection and difference) to two solids may yield lower dimensional components. For example, the intersection of two cubic solids sharing a common face is that face which is a 2-dimensional object. The regularized Boolean operators are used to remove these lower dimensional components by taking the interior of the result and then wrapping it up with its closure. ∎

3. [**25 points**] Suppose a polyhedron's faces are *all* triangles. The *star* of a given vertex consists of *all* triangles that are adjacent to this vertex listed in clockwise or counter-clockwise order. The *link* of the given vertex consists of all edges of the triangles in the star that are not adjacent to the given vertex. In the figure on the next page, the given vertex is the yellow square. Its star consists of all triangles whose vertices are the yellow square and its two adjacent circles, and the link is marked with thick light color edges.

   Suppose you have a winged-edge data structure of such a polyhedron. Design an algorithm that accepts a vertex and reports the star and link of this vertex in clockwise or counter clockwise order.

   Solution: If you did Exercise 2 correctly, this is a simple problem; otherwise, it could be a little difficult.



procedure *PrintStar*(v: **vertex**)
begin
  $e = e_{\text{original}}$ = Vertex_table[v];
  do
    if v = Edge_table[e].start **then**   // use right face
      next = Edge_table[e].right_predecessor;
      vertex = Edge_table[e].end;
      edge = Edge_table[e].right_successor;
    else   // use left face
      next = Edge_table[e].left_predecessor;
      vertex = Edge_table[e].start;
      edge = Edge_table[e].left_successor;
    end if
    print *vertex* and *edge*
    e = *next*;
  while ($e \neq e_{\text{original}}$)
end

The idea of this algorithm is quite simple. We go through every triangle in clockwise order and print out the opposite vertex of v. The initial edge is saved to e_{original} and e, and use e for working. For each e, if the given vertex v is its start vertex, we take the right face and the vertex on e that is different from v is in the link. In the algorithm above, the other vertex is saved to *vertex*. The next edge of e of the visited triangle, *edge*, is an edge of the link. To move to the next edge in clockwise order, we take the right predecessor of e. If v is the end vertex of e, we use the left face. After the **if-then-else-end if**, we print out *vertex* and *edge*, set e to *next*. This process continues until the next edge e is equal to the initial edge e_{original}. ∎

4. **Bézier Curves**
Suppose a Bézier curve $\mathbf{C}(u)$ is defined by the following four control points in the xy-plane: $\mathbf{P}_0 = (-2, 0)$, $\mathbf{P}_1 = (-2, 4)$, $\mathbf{P}_2 = (2, 4)$ and $\mathbf{P}_3 = (2, 0)$. Do the following problems:

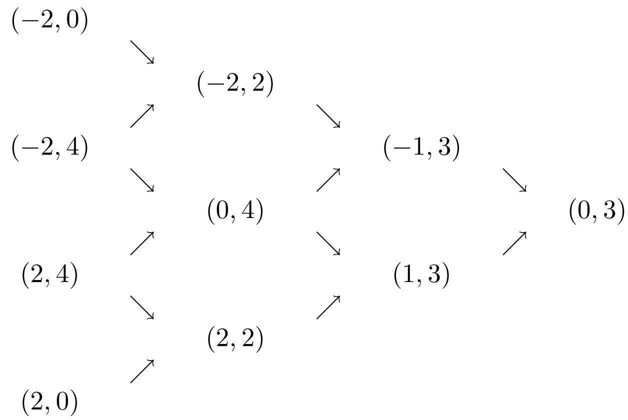
   (a) [**5 points**] What is the degree of $\mathbf{C}(u)$?
   Solution: The degree is equal to the number of control points minus 1. Therefore, the degree of the given Bézier curve is 3. ∎

   (b) [**10 points**] A Bézier curve $\mathbf{C}(u)$ defined by three control points $\mathbf{P}_0$, $\mathbf{P}_1$ and $\mathbf{P}_2$ can only be a parabola. Why? **Elaborate your answer in some details.**
   Solution: It was discussed in class that polynomials (in parametric form) can only represent parabolas, and circles, ellipses and hyperbolas require the use of rational form. Since Bézier

curves are parametric polynomial curves, they can only be parabolas. ∎

(c) **[15 points]** Compute $\mathbf{C}(1/2)$ with de Casteljau's algorithm. **You should show all computation steps and a diagram showing the de Casteljau net. You will receive <u>no</u> credit if you only provide a point and/or a diagram.**

<u>Solution</u>: Since $u = 0.5$, $1 - u = 0.5$. The following shows the computation steps:



Draw the diagram for yourself. ∎

(d) **[10 points]** Divide the curve at $\mathbf{C}(1/2)$, and list the control points of each curve segment in <u>*correct*</u> order.

<u>Solution</u>: From the above diagram, the "left" curve segment on [0,0.5] is a Bézier curve of degree 3 defined by control points $(-2, 0)$, $(-2, 2)$, $(-1, 3)$ and $(0, 3)$, and the "right" curve segment on [0.5,1] is a Bézier curve of degree 3 defined by control points $(0, 3)$, $(1, 3)$, $(2, 2)$ and $(2, 0)$. ∎

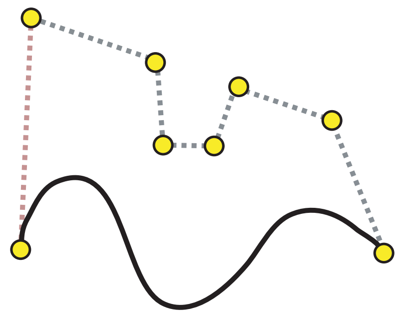
(e) **[15 points]** Increase the degree of the original curve by one. Draw a diagram that shows the original and degree-elevated control polygons. **You will receive <u>no</u> credit if you only provide a diagram.**

<u>Solution</u>: The new curve has degree 4 and control points $\mathbf{Q}_0$, $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$ and $\mathbf{Q}_4$, where $\mathbf{Q}_0 = \mathbf{P}_0$, $\mathbf{Q}_4 = \mathbf{P}_3$, and

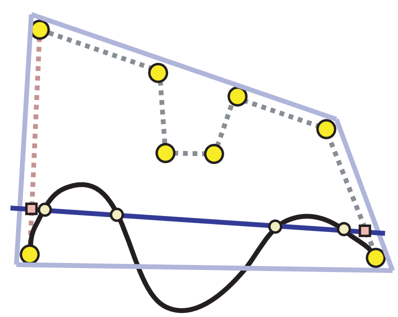
$$\begin{aligned}
\mathbf{Q}_1 &= \frac{1}{3+1}\mathbf{P}_0 + \left(1 - \frac{1}{3+1}\right)\mathbf{P}_1 = (-2, 3) \\
\mathbf{Q}_2 &= \frac{2}{3+1}\mathbf{P}_1 + \left(1 - \frac{2}{3+1}\right)\mathbf{P}_2 = (0, 4) \\
\mathbf{Q}_3 &= \frac{3}{3+1}\mathbf{P}_2 + \left(1 - \frac{3}{3+1}\right)\mathbf{P}_3 = (2, 3)
\end{aligned}$$

Draw a diagram for yourself. ∎

(f) **[10 points]** A programmer wrote a program for drawing Bézier curves. The following figure shows one of his result. This programmer is very certain that the curve is tangent to the first and last legs at the first and last control points, respectively. However, he suspects his program still has problems. Find all problems you can in this figure. You have to **(1)** identify each problem *explicitly*, and **(2)** provide a convincing explanation. **You will receive <u>no</u> credit if you only provide an answer.**



Solution: Consider the straight line shown in the figure below. This line intersects the control polygon and the Bézier curve in two (square) and four (circular) points. This violates the *variation diminishing property*, and, as a result, this Bézier curve is incorrect. Moreover, since a portion of the curve is not in the convex hull of the control points, this violates the convex hull property. ■



(g) **[25 points]** Suppose we have two Bézier curves, $\mathbf{C}_1(u)$ and $\mathbf{C}_2(u)$, where $\mathbf{C}_1(u)$ is defined by control points $\mathbf{P}_0 = (0, -2)$, $\mathbf{P}_1 = (-2, -2)$, $\mathbf{P}_2 = (-2, 0)$ and $\mathbf{P}_3 = (0, 0)$, and $\mathbf{C}_2(v)$ is defined by control points $\mathbf{Q}_0 = \mathbf{P}_3 = (0, 0)$, $\mathbf{Q}_1 = (2, 0)$, $\mathbf{Q}_2 = (2, 3)$ and $\mathbf{Q}_3 = (0, 3)$, and $u, v \in [0, 1]$. Discuss the continuity at $\mathbf{P}_3 = \mathbf{Q}_0 = (0, 0)$. More precisely, are these two curves C^1-, C^2-, G^1-, G^2- and curvature continuous at $(0, 0)$? **You need to show your calculations. Without explicit calculation, you will receive no credit.**

Solution: Consider curve $\mathbf{C}_1(u)$ first. It has a degree of $p = 3$, and its first and second derivative curves are

$$\begin{aligned}
\mathbf{C}_1(u) &= \sum_{i=0}^{3} B_{3,i}(u)\mathbf{P}_i \\
\mathbf{C}_1'(u) &= \sum_{i=0}^{2} B_{2,i}(u)\,[3(\mathbf{P}_{i+1} - \mathbf{P}_i)] \\
\mathbf{C}_1''(u) &= \sum_{i=0}^{1} B_{1,i}(u)\,[6(\mathbf{P}_{i+2} - 2\mathbf{P}_{i+1} + \mathbf{P}_i)]
\end{aligned}$$

Since the first and second derivatives of $\mathbf{C}_1(u)$ at $u = 1$ are $\mathbf{C}_1'(1)$ and $\mathbf{C}_1''(1)$, respectively, we have $\mathbf{C}_1(1) = \mathbf{P}_3 = (0, 0)$, $\mathbf{C}_1'(1) = (6, 0)$ and $\mathbf{C}_1''(1) = (12, -12)$.

Now consider curve $\mathbf{C}_2(u)$. It has a degree of $p = 3$, and its first and second derivative curves are

$$\mathbf{C}_2(u) = \sum_{i=0}^{3} B_{3,i}(u)\mathbf{Q}_i$$

$$\begin{aligned} \mathbf{C}_2'(u) &= \sum_{i=0}^{2} B_{2,i}(u)\,[3(\mathbf{Q}_{i+1} - \mathbf{Q}_i)] \\ \mathbf{C}_2''(u) &= \sum_{i=0}^{1} B_{1,i}(u)\,[6(\mathbf{Q}_{i+2} - 2\mathbf{Q}_{i+1} + \mathbf{Q}_i)] \end{aligned}$$

Since the first and second derivatives of $\mathbf{C}_2(u)$ at $u = 0$ are $\mathbf{C}_2'(0)$ and $\mathbf{C}_2''(0)$, respectively, we have $\mathbf{C}_2(0) = \mathbf{P}_3 = (0,0)$, $\mathbf{C}_2'(0) = (6,0)$ and $\mathbf{C}_2''(0) = (-12,18)$.

- Since $\mathbf{C}_1(1) = \mathbf{C}_2(0) = (0,0)$, $\mathbf{C}_1(u)$ and $\mathbf{C}_2(u)$ are C^0-continuous at $(0,0)$.
- Since $\mathbf{C}_1'(1) = \mathbf{C}_2'(0) = (6,0)$, $\mathbf{C}_1(u)$ and $\mathbf{C}_2(u)$ are C^1-continuous at $(0,0)$. They are also G^1-continuous at $(0,0)$.
- Since $\mathbf{C}_1''(1) = (12,-12) \neq \mathbf{C}_2''(0) = (-12,18)$, $\mathbf{C}_1(u)$ and $\mathbf{C}_2(u)$ are *not* C^2-continuous at $(0,0)$.
- Since $\mathbf{C}_1''(1) - \mathbf{C}_2''(0) = (24,-30)$, which is not parallel to the tangent vector $(6,0)$ at $(0,0)$, $\mathbf{C}_1(u)$ and $\mathbf{C}_2(u)$ are *not* G^2-continuous at $(0,0)$.
- The curvature of $\mathbf{C}_1(u)$ at $(0,0)$ is computed as

$$\kappa_1(1) = \frac{|\mathbf{C}_1'(1) \times \mathbf{C}_1''|}{|\mathbf{C}_1'(1)|^3} = \frac{|(6,0,0) \times (12,-12,0)|}{|(6,0,0)|^3} = \frac{1}{3}$$

  and the curvature of $\mathbf{C}_2(u)$ at $u = 0$ is computed as follows:

$$\kappa_2(0) = \frac{|\mathbf{C}_2'(1) \times \mathbf{C}_2''|}{|\mathbf{C}_2'(1)|^3} = \frac{|(6,0,0) \times (-12,18,0)|}{|(6,0,0)|^3} = \frac{1}{2}$$

  Since $\kappa_1(1) \neq \kappa_2(0)$, $\mathbf{C}_1(u)$ and $\mathbf{C}_2(u)$ are *not* curvature continuous at $(0,0)$.

In summary, $\mathbf{C}_1(u)$ and $\mathbf{C}_2(u)$ are C^0-, G^1- and C^1- continuous, but not G^2-, C^2- and curvature continuous at $(0,0)$. ∎