CS3621 Exercise 3 Solution (Fall 2005)

1. Consider the following two circular arcs joining at the origin: $\mathbf{f}(u) = (\cos(u+\pi/2), -(1+\sin(u+\pi/2)), 0)$ and $\mathbf{g}(v) = (-\cos(v+\pi/2), 0, 1-\sin(v+\pi/2))$, where both u and v are in the range of 0 and π. Note that circular arcs $\mathbf{f}(u)$ and $\mathbf{g}(v)$ lie on the xy- and xz-coordinate planes, respectively. Analyze the continuity at the origin. More precisely, are $\mathbf{f}(u)$ and $\mathbf{g}(v)$ C^1, C^2, G^1 or G^2 at the origin $\mathbf{f}(\pi) = \mathbf{g}(0) = (0, 0, 0)$? Are they curvature continuous?

   Solution: Since $\sin(u + \frac{\pi}{2}) = \cos(u)$ and $\cos(u + \frac{\pi}{2}) = -\sin(u)$, the given functions can be rewritten as

$$\mathbf{f}(u) = \langle -\sin(u), -(1+\cos(u)), 0\rangle \qquad \text{and} \qquad \mathbf{g}(v) = \langle \sin(v), 0, 1-\cos(v)\rangle$$

   Thus, we have

$$\begin{aligned}
\mathbf{f}'(u) &= \langle -\cos(u), \sin(u), 0\rangle \\
\mathbf{f}''(u) &= \langle \sin(u), \cos(u), 0\rangle \\
\mathbf{g}'(v) &= \langle \cos(v), 0, \sin(v)\rangle \\
\mathbf{g}''(v) &= \langle -\sin(v), 0, \cos(v)\rangle
\end{aligned}$$

   Since $\mathbf{f}(\pi) = \mathbf{g}(0) = \langle 0, 0, 0\rangle$, $\mathbf{f}(u)$ and $\mathbf{g}(v)$ are C^0 continuous at the origin.

   Since $\mathbf{f}'(\pi) = \mathbf{g}'(0) = \langle 1, 0, 0\rangle$, $\mathbf{f}(u)$ and $\mathbf{g}(v)$ are C^1 continuous at the origin.

   Since $\mathbf{f}''(\pi) = \langle 0, -1, 0\rangle$ and $\mathbf{g}''(0) = \langle 0, 0, 1\rangle$, $\mathbf{f}(u)$ and $\mathbf{g}(v)$ are *not* C^2 continuous at the origin.

   Since $\mathbf{f}(u)$ and $\mathbf{g}(v)$ are C^1 at the origin, they are G^1 at the origin.

   Since $\mathbf{f}(u)$ and $\mathbf{g}(v)$ are C^1 at the origin, their tangent line at the origin is the x-axis. Since $\mathbf{f}''(\pi) = \langle 0, -1, 0\rangle$ and $\mathbf{g}''(0) = \langle 0, 0, 1\rangle$, we have $\mathbf{f}''(\pi) - \mathbf{g}''(0) = \langle 0, -1, -1\rangle$. Since this vector is not parallel to the tangent line, $\mathbf{f}(u)$ and $\mathbf{g}(v)$ are *not* G^2 continuous at the origin.

   The following computes the curvatures of $\mathbf{f}(u)$ and $\mathbf{g}(v)$:

$$\begin{aligned}
\mathbf{f}'(u) \times \mathbf{f}''(u) &= \langle 0, 0, -1\rangle \\
|\mathbf{f}'(u)| &= 1 \\
\kappa_{\mathbf{f}}(u) &= \frac{|\mathbf{f}'(u) \times \mathbf{f}''(u)|}{|\mathbf{f}'(u)|} = 1 \\
\mathbf{g}'(v) \times \mathbf{g}''(v) &= \langle 0, -1, 0\rangle \\
|\mathbf{g}'(v)| &= 1 \\
\kappa_{\mathbf{g}}(v) &= \frac{|\mathbf{g}'(v) \times \mathbf{g}''(v)|}{|\mathbf{g}'(v)|} = 1
\end{aligned}$$

   Since $\kappa_{\mathbf{f}}(u) = \kappa_{\mathbf{g}}(v) = 1$, $\mathbf{f}(u)$ and $\mathbf{g}(v)$ are curvature continuous at the origin. ∎

2. Given six control points on the xy-plane $(0, -2)$, $(-2, -2)$, $(-2, 2)$, $(2, 2)$, $(2, -2)$ and $(0, -2)$, do the following:

   (a) Compute the partition of unity at $u = 0.5$.

   Solution: The following are Bézier basis functions at $u = 0.5$:

$$B_{5,0}(u) = \frac{5!}{0!(5-0)!} u^0 (1-u)^{5-0} = 1 \cdot 0.5^5 = 0.03125$$

$$
\begin{aligned}
B_{5,1}(u) &= \frac{5!}{1!(5-1)!}u^1(1-u)^{5-1} = 5 \cdot 0.5^5 = 0.15625 \\
B_{5,2}(u) &= \frac{5!}{2!(5-2)!}u^2(1-u)^{5-2} = 10 \cdot 0.5^5 = 0.3125 \\
B_{5,3}(u) &= \frac{5!}{3!(5-3)!}u^3(1-u)^{5-3} = 10 \cdot 0.5^5 = 0.3125 \\
B_{5,4}(u) &= \frac{5!}{4!(5-4)!}u^4(1-u)^{5-4} = 5 \cdot 0.5^5 = 0.15625 \\
B_{5,5}(u) &= \frac{5!}{5!(5-5)!}u^5(1-u)^{5-5} = 1 \cdot 0.5^5 = 0.03125
\end{aligned}
$$

Verify that the sum of these six terms is 1. ∎

(b) Use de Casteljau's algorithm to find the point on the curve that corresponds to $u = 0.5$.
Solution:

$(0,-2)$										
	$\searrow$									
		$(-1,-2))$								
	$\nearrow$		$\searrow$							
$(-2,-2)$				$(-1.5,-1)$						
	$\searrow$		$\nearrow$		$\searrow$					
		$(-2,0)$				$(-1.25,0)$				
	$\nearrow$		$\searrow$		$\nearrow$		$\searrow$			
$(-2,2)$				$(-1,1)$				$(-0.625,0.5)$		
	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$		$\searrow$	
		$(0,2)$				$(0,1)$				$(0,0.5)$
	$\nearrow$		$\searrow$		$\nearrow$		$\searrow$		$\nearrow$	
$(2,2)$				$(1,1)$				$(0.625,0.5)$		
	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$			
		$(2,0)$				$(1.25,0)$				
	$\nearrow$		$\searrow$		$\nearrow$					
$(2,-2)$				$(1.5,-1)$						
	$\searrow$		$\nearrow$							
		$(1,-2)$								
	$\nearrow$									
$(0,-2)$										

Thus, the point curve corresponding to $u = 0.5$ is $\mathbf{C}(0.5) = (0, 0.5)$. ∎

(c) Subdivide the Bzier curve at $u = 0.5$ and list the control points of the resulting curve segments.
Solution: The first curve segment, defined on [0,0.5], has control points $(0,-2)$, $(-1,-2)$, $(-1.5,-1)$, $(-1.25,0)$, $(-0.625,0.5)$ and $(0,0.5)$, and the second segment, defined on [0.5,1], has control points $(0,0.5)$, $(0.625,0.5)$, $(1.25,0)$, $(1.5,-1)$, $(1,-2)$ and $(0,-2)$. Please note the order or the control points of the second curve segment. ∎

(d) Increase the degree of this curve to six and list the new set of control points.
Solution: The new set of control points is computed as follows:

$$\mathbf{Q}_0 = \mathbf{P}_0$$

$$
\begin{aligned}
\mathbf{Q}_1 &= \frac{1}{6}\mathbf{P}_0 + \frac{5}{6}\mathbf{P}_1 = \left(-\frac{5}{3}, -2\right) \\
\mathbf{Q}_2 &= \frac{2}{6}\mathbf{P}_1 + \frac{4}{6}\mathbf{P}_2 = \left(-2, \frac{2}{3}\right) \\
\mathbf{Q}_3 &= \frac{3}{6}\mathbf{P}_2 + \frac{3}{6}\mathbf{P}_3 = (0, 2) \\
\mathbf{Q}_4 &= \frac{4}{6}\mathbf{P}_3 + \frac{1}{6}\mathbf{P}_4 = \left(2, \frac{2}{3}\right) \\
\mathbf{Q}_5 &= \frac{5}{6}\mathbf{P}_4 + \frac{1}{6}\mathbf{P}_5 = \left(\frac{5}{3}, -2\right) \\
\mathbf{Q}_6 &= \mathbf{P}_5
\end{aligned}
$$

(e) The starting point and the ending point of this curve are the same $(0, -2)$. Discuss the continuity issue at the joining point. More precisely, is it C^1, C^2, G^1, G^2 or curvature continuous?

Solution: Since the join point is $\mathbf{P}_0 = \mathbf{P}_5 = (0, -2)$, what we need are the first and second derivatives at $\mathbf{P}_0$ and $\mathbf{P}_5$. First, we have $\mathbf{C}(0) = \mathbf{C}(1) = (0, -2)$ because a Bézier curve passes through the first and last control points. By the same reason, $\mathbf{C}'(0) = 5(\mathbf{P}_1 - \mathbf{P}_0) = (-10, 0)$ and $\mathbf{C}'(1) = 5(\mathbf{P}_5 - \mathbf{P}_4) = (-10, 0)$. The second derivative at 0 and 1 are $\mathbf{C}''(0) = 5 \cdot 4 \cdot (\mathbf{P}_2 - 2\mathbf{P}_1 + \mathbf{P}_0) = (40, 80)$ and $\mathbf{C}''(1) = 5 \cdot 4 \cdot (\mathbf{P}_5 - 2\mathbf{P}_4 + \mathbf{P}_3) = (-40, 80)$. Now, we have the following:

- C^0-continuous: **YES** because $\mathbf{C}(0) = \mathbf{C}(1) = \mathbf{P}_0 = \mathbf{P}_1 = (0, -2)$.
- C^1- and G^1- continuous: **YES** because $\mathbf{C}'(0) = \mathbf{C}'(1) = (-10, 0)$. It is G^1 because C^1 implies G^1.
- C^2-continuous: **NO** because $\mathbf{C}''(0) = (40, 80) \neq \mathbf{C}''(1) = (-40, 80)$.
- G^2-continuous: **YES**. The curve goes from the right side, crosses $\mathbf{P}_5$, and enters the left side. In other words, the "left" curve is the curve segment on $[u, 1]$ where u is close to 1, and the "right" curve is the curve segment on $[0, v]$ where v is close to 0. Now, the difference of the second derivative of the "left" curve and the second derivative of the "right" curve is $\mathbf{C}''(1) - \mathbf{C}''(0) = (-80, 0)$. Since this vector is parallel to the tangent vector at $\mathbf{C}(0) = \mathbf{C}(1)$, the curve is of G^2-continuous at $(0, -2)$.
- Curvature Continuous: **YES**. The curvature at $\mathbf{C}(0) = \mathbf{P}_0$ is computed as follows:

$$\kappa_{\mathbf{C}(0)} = \frac{|\mathbf{C}'(0) \times \mathbf{C}''(0)|}{|\mathbf{C}'(0)|^3} = \frac{|\langle -10, 0, 0\rangle \times \langle 40, 80, 0\rangle|}{|\langle -10, 0, 0\rangle|^3} = 0.8$$

The curvature at $\mathbf{C}(1) = \mathbf{P}_5$ is

$$\kappa_{\mathbf{C}(1)} = \frac{|\mathbf{C}'(1) \times \mathbf{C}''(1)|}{|\mathbf{C}'(1)|^3} = \frac{|\langle -10, 0, 0\rangle \times \langle -40, 80, 0\rangle|}{|\langle -10, 0, 0\rangle|^3} = 0.8$$

Since $\kappa_{\mathbf{C}(0)} = \kappa_{\mathbf{C}(1)}$, the curve is curvature continuous at $(0, -2)$.