CS3621 Exercise 4 Solutions (Fall 2005)

1. Suppose the knot vector is $U = \{0.0, 0.2, 0.5, 0.5, 0.8, 1\}$. Compute the basis function $N_{2,2}(u)$ and sketch its graph. Please also indicate the non-zero domain of this basis function.

   Answer: Recall the following Cox-de Boor formula:

$$N_{i,0} = \begin{cases} 1 & u \in [u_i, u_{i+1}) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,p} = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u)$$

   $N_{2,2}(u)$ is non-zero on $[u_2, u_5)$, where $i = p = 2$. Since the knot vector is

u_0	u_1	u_2	u_3	u_4	u_5
0	0.2	0.5	0.5	0.8	1

   $N_{2,2}(u)$ is non-zero on $[0.5, 1)$. Since $N_{2,0}(u)$ is non-zero on $[u_2, u_3) = [0.5, 0.5)$ and since $[0.5, 0.5)$ contains no point, $N_{2,0}(u)$ is zero everywhere and can be ignored. Note that $N_{3,0}(u)$ and $N_{4,0}(u)$ are non-zero on $[u_3, u_4) = [0.5, 0.8)$ and $[u_4, u_5) = [0.8, 1)$, respectively.

   Let us calculate $N_{2,1}(u)$. Since $N_{2,1}(u)$ uses $N_{2,0}(u)$ and $N_{3,0}(u)$ and since $N_{2,0}(u)$ is zero everywhere, $N_{2,1}(u)$ is non-zero on $N_{3,0}(u)$'s domain $[u_3, u_4) = [0.5, 0.8)$. By Cox-de Boor formula, we have

$$N_{2,1}(u) = \frac{u - u_2}{u_3 - u_2} N_{2,0}(u) + \frac{u_4 - u}{u_4 - u_3} N_{3,0}(u) = \frac{u_4 - u}{u_4 - u_3} N_{3,0}(u) = \frac{0.8 - u}{0.8 - 0.5} \cdot 1 = \frac{1}{3}(8 - 10u)$$

   Now consider $N_{3,1}(u)$. Since $N_{3,1}(u)$ is non-zero on $[u_3, u_5) = [u_3, u_4) \cup [u_4, u_5) = [0.5, 0.8) \cup [0.8, 1)$, we need to consider each knot span separately:

   - Knot span $[u_3, u_4) = [0.5, 0.8)$:
     Since only $N_{3,0}(u)$ is non-zero on $[0.5, 0.8)$, we have

$$N_{3,1}(u) = \frac{u - u_3}{u_4 - u_3} N_{3,0}(u) + \frac{u_5 - u}{u_5 - u_4} N_{4,0}(u) = \frac{u - u_3}{u_4 - u_3} N_{3,0} = \frac{u - 0.5}{0.8 - 0.5} = -\frac{1}{3}(5 - 10u)$$

   - Knot span $[u_4, u_5) = [0.8, 1)$:
     Since only $N_{4,0}(u)$ is non-zero on $[0.8, 1)$, we have

$$N_{3,1}(u) = \frac{u - u_3}{u_4 - u_3} N_{3,0}(u) + \frac{u_5 - u}{u_5 - u_4} N_{4,0}(u) = \frac{u_5 - u}{u_5 - u_4} N_{4,0}(u) = \frac{1 - u}{1 - 0.8} = 5(1 - u)$$

   Consequently, $N_{3,1}(u)$ is

$$N_{3,1}(u) = \begin{cases} -\frac{1}{3}(5 - 10u) & u \in [0.5, 0.8) \\ 5(1 - u) & u \in [0.8, 1) \\ 0 & \text{otherwise} \end{cases}$$

   With $N_{2,1}(u)$ and $N_{3,1}(u)$ in hand, we can compute $N_{2,2}(u)$. Again, we need to do our calculation on each knot span.

- Knot span $[u_3, u_4) = [0.5, 0.8)$:
  Since $N_{2,1}(u)$ and $N_{3,1}(u)$ are both non-zero on $[u_3, u_4) = [0.5, 0.8)$, we have

$$\begin{aligned} N_{2,2}(u) &= \frac{u - u_2}{u_4 - u_2} N_{2,1}(u) + \frac{u_5 - u}{u_5 - u_3} N_{3,1}(u) \\ &= \frac{u - 0.5}{0.8 - 0.5}\left(\frac{1}{3}(8 - 10u)\right) + \frac{1 - u}{1 - 0.5}\left(-\frac{1}{3}(5 - 10u)\right) \\ &= \frac{1}{9}\left(-70 + 220u - 160u^2\right) \end{aligned}$$

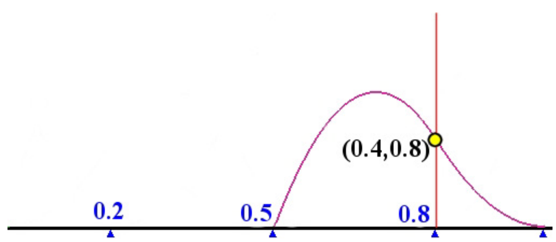
- Knot span $[u_4, u_5) = [0.8, 1)$:
  Since only $N_{3,1}(u)$ is non-zero on $[0.8, 1)$, we have

$$N_{2,2}(u) = \frac{u - u_2}{u_4 - u_2} N_{2,1}(u) + \frac{u_5 - u}{u_5 - u_3} N_{3,1}(u) = \frac{u_5 - u}{u_5 - u_3} N_{3,1}(u) = \frac{1 - u}{1 - 0.5}\left(5(1 - u)\right) = 10(1 - u)^2$$

Therefore, $N_{2,2}(u)$ is the following:

$$N_{2,2}(u) = \begin{cases} \frac{1}{9}\left(-70 + 220u - 160u^2\right) & u \in [0.5, 0.8) \\ 10(1 - u)^2 & u \in [0.8, 1) \\ 0 & \text{otherwise} \end{cases}$$

The following is the graph of $N_{2,2}(u)$:



2. Suppose a clamped B-spline curve of degree 3 is defined by control points $(-2, 0)$, $(-2, 2)$, $(-1, 2)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, $(1, 2)$, $(2, 2)$ and $(2, 0)$ and knots 0 (multiplicity 4), 1/6, 1/3, 1/2, 2/3, 5/6 and 1 (multiplicity 4). **Insert a new knot at 0.4 twice**. You should show all detailed computation of both insertions, including all computations, diagrams, all existing and new control points, and the way that corners are cut.

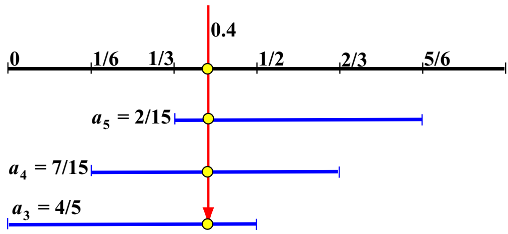
   Answer: Based on the given information, we have the following control points:

$\mathbf{P}_0$	$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_5$	$\mathbf{P}_6$	$\mathbf{P}_7$	$\mathbf{P}_8$
$(-2, 0)$	$(-2, 2)$	$(-1, 2)$	$(-1, 1)$	$(0, 0)$	$(1, 1)$	$(1, 2)$	$(2, 2)$	$(2, 0)$

and the following knot vector:

$u_0 = u_1 = u_2 = u_3$	u_4	u_5	u_6	u_7	u_8	$u_9 = u_{10} = u_{11} = u_{12}$
0	1/6	1/3	1/2	2/3	5/6	1

Since $u = 0.4 \in [u_5, u_6) = [1/3, 1/2)$, the affected control points are $\mathbf{P}_5$, $\mathbf{P}_4$, $\mathbf{P}_3$ and $\mathbf{P}_2$. Note that the degree is $p = 3$. The knot insertion diagram is



The new control points are

$$\begin{aligned}\mathbf{Q}_5 &= \left(1-\frac{2}{15}\right)\mathbf{P}_4 + \frac{2}{15}\mathbf{P}_5 = \left(\frac{2}{15}, \frac{2}{15}\right)\\ \mathbf{Q}_4 &= \left(1-\frac{7}{15}\right)\mathbf{P}3 + \frac{7}{15}\mathbf{P}_4 = \left(-\frac{8}{15}, \frac{8}{15}\right)\\ \mathbf{Q}_3 &= \left(1-\frac{4}{5}\right)\mathbf{P}_2 + \frac{4}{5}\mathbf{P}_3 = \left(-1, \frac{6}{5}\right)\end{aligned}$$

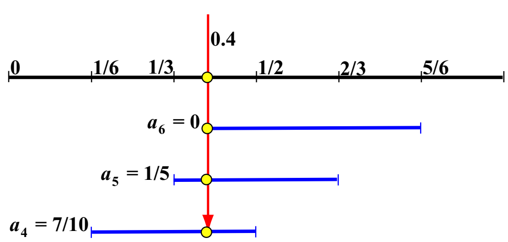
Therefore, after the first insertion, the control points are

$\mathbf{P}_0$	$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_5$	$\mathbf{P}_6$	$\mathbf{P}_7$	$\mathbf{P}_8$	$\mathbf{P}_9$
$(-2,0)$	$(-2,2)$	$(-1,2)$	$(-1,6/5)$	$(-8/15,8/15)$	$(2/15,2/15)$	$(1,1)$	$(1,2)$	$(2,2)$	$(2,0)$

and the following knot vector:

$u_0=u_1=u_2=u_3$	u_4	u_5	u_6	u_7	u_8	u_9	$u_{10}=u_{11}=u_{12}=u_{13}$
0	1/6	1/3	0.4	1/2	2/3	5/6	1

For the second insertion, we have $u = 0.4 \in [u_6, u_7)$ and the affected control points are $\mathbf{P}_6$, $\mathbf{P}_5$, $\mathbf{P}_4$ and $\mathbf{P}_3$. The knot insertion diagram gives:



Therefore, the new control points are

$$\begin{aligned}\mathbf{Q}_6 &= (1-0)\mathbf{P}_5 + 0\mathbf{P}_6 = \mathbf{P}_5\\ \mathbf{Q}_5 &= \left(1-\frac{1}{5}\right)\mathbf{P}_4 + \frac{1}{5}\mathbf{P}_5 = \left(-\frac{2}{5}, \frac{34}{15}\right)\\ \mathbf{Q}_4 &= \left(1-\frac{7}{10}\right)\mathbf{P}_3 + \frac{7}{10}\mathbf{P}_4\end{aligned}$$

In summary, after two insertions of $u = 0.4$, the new set of control points are

$\mathbf{P}_0$	$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_5$	$\mathbf{P}_6$	$\mathbf{P}_7$	$\mathbf{P}_8$	$\mathbf{P}_9$	$\mathbf{P}_{10}$
$(-2,0)$	$(-2,2)$	$(-1,2)$	$\left(-1,\frac{6}{5}\right)$	$\left(-\frac{101}{150},\frac{110}{150}\right)$	$\left(-\frac{2}{5},\frac{34}{75}\right)$	$\left(\frac{2}{15},\frac{2}{15}\right)$	$(1,1)$	$(1,2)$	$(2,2)$	$(2,0)$

and the new knot vector is

$u_0 = u_1 = u_2 = u_3$	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	$u_{11} = u_{12} = u_{13} = u_{14}$
0	1/6	1/3	0.4	0.4	1/2	2/3	5/6	1

3. Suppose a clamped B-spline curve of degree 3 is defined by $(-2, 0)$, $(-2, 2)$, $(2, 2)$ and $(2, 0)$ and knots 0 (multiplicity 4) and 1 (multiplicity 4). Provide the computation, in triangular form, with all the detailed steps using de Boor's algorithm and show that the result is exactly the same as that of de Casteljau's algorithm. **You will receive no credit if you do not show the details.**

<u>Answer</u>: From the question, we have degree $p = 3$, control points

$\mathbf{P}_0$	$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$
$(-2, 0)$	$(-2, 2)$	$(2, 2)$	$(2, 0)$

and knot vector

$u_0 = u_1 = u_2 = u_3$	$u_4 = u_5 = u_6 = u_7$
0	1

The first insertion of u gives $u \in [u_3, u_4)$. The affected control points are $\mathbf{P}_3$, $\mathbf{P}_2$, $\mathbf{P}_1$ and $\mathbf{P}_0$, and $a_3 = (u - u_3)/(u_6 - u_3) = u$, $a_2 = (u - u_2)/(u_5 - u_2) = u$ and $a_1 = (u - u_1)/(u_4 - u_1) = u$. Therefore, the new control points are $\mathbf{Q}_3 = (1 - u)\mathbf{P}_2 + u\mathbf{P}_3$, $\mathbf{Q}_2 = (1 - u)\mathbf{P}_1 + u\mathbf{P}_2$ and $\mathbf{Q}_1 = (1 - u)\mathbf{P}_0 + u\mathbf{P}_1$. If we arrange $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ on the first column and $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{Q}_3$ on the second as follows:

$$
\begin{array}{ccc}
\mathbf{P}_0 & & \\
 & \searrow & \\
 & & \mathbf{Q}_1 \\
 & \nearrow & \\
\mathbf{P}_1 & & \\
 & \searrow & \\
 & & \mathbf{Q}_2 \\
 & \nearrow & \\
\mathbf{P}_2 & & \\
 & \searrow & \\
 & & \mathbf{Q}_3 \\
 & \nearrow & \\
\mathbf{P}_3 & &
\end{array}
$$

we see that $\searrow$ and $\nearrow$ are assigned with $1 - u$ and u, respectively. Therefore, the result of the first insertion is identical to that of de Casteljau's algorithm.

After the first insertion, the new control points are

$\mathbf{P}_0$	$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$
$\mathbf{P}_0$	$\mathbf{Q}_1$	$\mathbf{Q}_2$	$\mathbf{Q}_3$	$\mathbf{P}_3$

and the knot vector is

$u_0 = u_1 = u_2 = u_3$	u_4	$u_5 = u_6 = u_7 = u_8$
0	u	1

The second insertion has $u \in [u_4, u_5)$ and the affected control points are $\mathbf{P}_3$, $\mathbf{Q}_3$, $\mathbf{Q}_2$ and $\mathbf{Q}_1$. Since $a_4 = (u - u_4)/(u_7 - u_4) = 0$, $a_3 = (u - u_3)/(u_6 - u_3) = u$ and $a_2 = (u - u_2)/(u_5 - u_2) = u$, the new control points are $\mathbf{R}_4 = \mathbf{Q}_3$, $\mathbf{R}_3 = (1-u)\mathbf{Q}_2 + u\mathbf{Q}_3$ and $\mathbf{R}_2 = (1-u)\mathbf{Q}_1 + u\mathbf{Q}_2$. If $\mathbf{R}_3$, $\mathbf{R}_2$ are added to the third column of the above diagram, we have

$$
\begin{array}{ccccc}
\mathbf{P}_0 & & & & \\
 & \searrow & & & \\
 & & \mathbf{Q}_1 & & \\
 & \nearrow & & \searrow & \\
\mathbf{P}_1 & & & & \mathbf{R}_2 \\
 & \searrow & & \nearrow & \\
 & & \mathbf{Q}_2 & & \\
 & \nearrow & & \searrow & \\
\mathbf{P}_2 & & & & \mathbf{R}_3 \\
 & \searrow & & \nearrow & \\
 & & \mathbf{Q}_3 & & \\
 & \nearrow & & & \\
\mathbf{P}_3 & & & &
\end{array}
$$

Again, we see that $\mathbf{R}_2$ and $\mathbf{R}_3$ are computed by assigning $\searrow$ and $\nearrow$ with $1-u$ and u, respectively. Therefore, the result of the second insertion is equal to the result on the third column of de Casteljau's algorithm.

The new set of control points is

$\mathbf{P}_0$	$\mathbf{P}_1$	$\mathbf{P}_2$	$\mathbf{P}_3$	$\mathbf{P}_4$	$\mathbf{P}_5$
$\mathbf{P}_0$	$\mathbf{Q}_1$	$\mathbf{R}_2$	$\mathbf{R}_3$	$\mathbf{Q}_3$	$\mathbf{P}_3$

and the new knot vector is

$u_0 = u_1 = u_2 = u_3$	$u_4 = u_5$	$u_6 = u_7 = u_8 = u_9$
0	u	1

The third insertion has $u \in [u_5, u_6)$. The affected control points are $\mathbf{P}_3$, $\mathbf{Q}_3$, $\mathbf{R}_3$ and $\mathbf{R}_2$, and $a_5 = (u - u_5)/(u_8 - u_5) = 0$, $a_4 = (u - u_4)/(u_7 - u_4) = 0$ and $a_3 = (u - u_3)/(u_6 - u_3) = u$. Therefore, the only new control point is

$$\mathbf{S} = (1-u)\mathbf{R}_2 + u\mathbf{R}_3$$

If $\mathbf{S}$ is place on the fourth column as follows:

$$
\begin{array}{ccccccc}
\mathbf{P}_0 & & & & & & \\
 & \searrow & & & & & \\
 & & \mathbf{Q}_1 & & & & \\
 & \nearrow & & \searrow & & & \\
\mathbf{P}_1 & & & & \mathbf{R}_2 & & \\
 & \searrow & & \nearrow & & \searrow & \\
 & & \mathbf{Q}_2 & & & & \mathbf{S} \\
 & \nearrow & & \searrow & & \nearrow & \\
\mathbf{P}_2 & & & & \mathbf{R}_3 & & \\
 & \searrow & & \nearrow & & & \\
 & & \mathbf{Q}_3 & & & & \\
 & \nearrow & & & & & \\
\mathbf{P}_3 & & & & & &
\end{array}
$$

the result of the third insertion is identical to the result of the fourth column of de Casteljau's algorithm. Consequently, in this particular case, the result of de Boor's algorithm and the result of de Casteljau's algorithm are the same.