1. Find the line defined by points (3,5) and (7, 14).

**Solution:** The line that contains two points \((a, b)\) and \((c, d)\) has an equation of

\[
\frac{y - b}{x - a} = \frac{d - b}{c - a}
\]

Since \(a = 3\) and \(c = 7\) and are not equal, the above equation is usable. Therefore, we have

\[
\frac{y - 5}{x - 3} = \frac{14 - 5}{7 - 3}
\]

Rearranging terms yields the answer to this question:

\[
9x - 4y - 7 = 0
\]

You can verify this answer by plugging \((3,5)\) and \((7,14)\) into the equation.

2. What is the distance from the origin to the line with equation \(3x + 4y + 6 = 0\)?

**Solution:** From the given equation, we have \(A = 3\), \(B = 4\) and \(C = 6\). Therefore, the distance from the origin to this line is

\[
\text{distance} = \frac{|C|}{\sqrt{A^2 + B^2}} = \frac{6}{\sqrt{3^2 + 4^2}} = \frac{6}{5} = 1.2
\]

3. Find the equation of the plane defined by base point \(B\) and normal vector \(n\). Verify your result using \(B = (3,4,5)\) and \(n = (1,1,1)\).

**Solution:** The answer should be \(X \cdot n - B \cdot n = 0\), where \(\cdot\) is the inner product. Plugging the values for \(B\) and \(n\), we have

\[
X \cdot (1, 1, 1) - \langle 3, 4, 5 \rangle \cdot (1, 1, 1) = 0
\]

Expanding and rearranging yields:

\[
x + y + z - 12 = 0
\]

4. What is the inner product and cross product of vectors \(u = \langle 1, 3, 5 \rangle\) and \(v = \langle -2, 0, 4 \rangle\). What is the length of the cross product? What is the angle between these two vectors?

**Solution:** The inner and cross products are

\[
\langle 1, 3, 5 \rangle \cdot \langle -2, 0, 4 \rangle = 1 \cdot (-2) + 3 \cdot 0 + 5 \cdot 4 = (-2) + 0 + 20 = 18
\]

\[
\langle 1, 3, 5 \rangle \times \langle -2, 0, 4 \rangle = \begin{vmatrix} 3 & 5 & 1 \\ 0 & 4 & -2 \\ 1 & 3 & 0 \end{vmatrix} = (12, -14, 6)
\]

Since \(|\langle 1, 3, 5 \rangle| = (1^2 + 3^2 + 5^2)^{1/2} = \sqrt{35}\) and \(|\langle -2, 0, 4 \rangle| = ((-2)^2 + 0^2 + 4^2)^{1/2} = \sqrt{20}\), the angle \(\theta\) between \(\langle 1, 3, 5 \rangle\) and \(\langle -2, 0, 4 \rangle\) is computed as follows:

\[
\cos(\theta) = \frac{\langle 1, 3, 5 \rangle \cdot \langle -2, 0, 4 \rangle}{|\langle 1, 3, 5 \rangle| \cdot |\langle -2, 0, 4 \rangle|} = \frac{18}{\sqrt{35} \sqrt{20}} \approx 0.68 \quad \text{and} \quad \theta = 47.13^\circ
\]

The length of \(\langle 1, 3, 5 \rangle \times \langle -2, 0, 4 \rangle\) is computed as

\[
|\langle 1, 3, 5 \rangle \times \langle -2, 0, 4 \rangle| = |\langle 12, -14, 6 \rangle| = \sqrt{12^2 + (-14)^2 + 6^2} = \sqrt{144 + 196 + 36} = \sqrt{376} \approx 19.39
\]
5. Suppose a general second degree curve \( Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \) represents a conic curve. Use the concept of the line at infinity to prove that the curve is an ellipse, a hyperbola, or a parabola, if and only if \( B^2 - AC < 0, \) \( B^2 - 4AC > 0, \) or \( B^2 - AC = 0. \) This problem looks difficult; but, it should be easy if you understand the concept of the line at infinity covered in class. Try it.

Solution: What we covered in class is: a conic is an ellipse, hyperbola or a parabola if it intersects the line at infinity in no point, two points, and one point (or tangent).

Converting the given equation to a homogeneous one yields

\[
Ax^2 + 2Bxy + Cy^2 + 2Dxw + 2Eyw + Fw^2 = 0
\]

Setting \( w = 0 \) means computing the intersection of the conic and the line at infinity. Thus, the intersection points are the solutions of \( Ax^2 + 2Bxy + Cy^2 = 0. \) Since all multiples of the same Euclidean point represent the same point in homogeneous coordinate, we could divide the whole equation by \( y^2 \) if \( y \neq 0; \) otherwise, divide the whole equation by \( x^2 \) if \( y = 0. \) Note that \( x \) and \( y \) cannot be both zero, because the point becomes \((0, 0, 0), \) which is not a legal point in homogeneous coordinate. Let us assume \( y \neq 0. \) Thus, we have

\[
A \left( \frac{x}{y} \right)^2 + 2B \left( \frac{x}{y} \right) + C = 0
\]

Now solving for \( x/y \) (rather than \( x \) and \( y \)) yields

\[
\frac{x}{y} = -B \pm \sqrt{B^2 - AC}
\]

Therefore, if \( B^2 - AC < 0 \) we have no real root and the conic is an ellipse; if \( B^2 - AC = 0, \) we have a double root and the conic is a parabola; and if \( B^2 - AC > 0, \) we have two real roots, and the conic is a hyperbola. □

6. Convert the following second degree equation to its equivalent matrix form:

\[
3x^2 - 5xy + 5y^2 + x + 6y + 9 = 0
\]

Solution: The answer is

\[
\begin{bmatrix}
3 & -5/2 & 1/2 \\
-5/2 & 5 & 3 \\
1/2 & 3 & 9
\end{bmatrix}
\]

7. What is the equation of the following conic curve in matrix form?

\[
\begin{bmatrix}
-1 & 4 & -3 \\
4 & 1 & 2 \\
-3 & 2 & 3
\end{bmatrix}
\]

Solution: The answer is

\[-x^2 + 8xy + y^2 - 6x + 4y + 3 = 0\]

8. Convert the following equations to their homogeneous form:

(a) \( x^2 + y^2 + 3z^2 - 2xy + 3xz + yz + 3x - 5y - 6z + 10 = 0 \)

Solution: Replacing \( x, \) \( y \) and \( z \) with \( x/w, y/w \) and \( z/w, \) respectively, we have

\[
\left( \frac{x}{w} \right)^2 + \left( \frac{y}{w} \right)^2 + 3 \left( \frac{z}{w} \right)^2 - 2 \left( \frac{x}{w} \right) \left( \frac{y}{w} \right) + 3 \left( \frac{x}{w} \right) \left( \frac{z}{w} \right) + \frac{y}{w} \left( \frac{z}{w} \right) + 3 \left( \frac{x}{w} \right) - 5 \frac{y}{w} - 6 \frac{z}{w} + 10 = 0
\]

Expanding the terms gives

\[
\frac{x^2}{w^2} + \frac{y^2}{w^2} + 3 \frac{z^2}{w^2} - 2 \frac{xy}{w^2} + 3 \frac{xz}{w^2} + \frac{yz}{w^2} + 3 \frac{x}{w} - 5 \frac{y}{w} - 6 \frac{z}{w} + 10 = 0
\]
9. Convert the following homogeneous equations to their non-homogeneous form:

(a) \( x^2 + y^2 + 3z^2 - 2xy + 3xz + yz + 3xw - 5yw - 6zw + 10w^2 = 0 \)

Solution: The degree of this homogeneous equation is 5. Thus, dividing the whole equation by \( w^2 \) gives
\[
\left( \frac{x}{w} \right)^2 + \frac{y}{w} \left( \frac{y}{w} \right)^2 - \frac{2x}{w} + \frac{3z}{w} \left( \frac{y}{w} \right)^2 + \frac{3x}{w} - \frac{5y}{w} - \frac{6z}{w} + 5 = 0
\]

(b) \( (x^2 + xy^2 - y^2)^2 - 4(x - y^2) - 1 = 0 \)

Solution: Replacing \( x, y \) and \( z \) with \( x/w, y/w \) and \( z/w \), respectively, we have
\[
\left( \frac{x^2}{w^2} + \frac{xy^2}{w^3} - \frac{y^2}{w^2} \right)^2 - 4 \left( \frac{x}{w} - \frac{y^2}{w^2} \right) - 1 = 0
\]

This equation can be rewritten as:
\[
\frac{x^2}{w^2} + \frac{xy^2}{w^3} - \frac{y^2}{w^2} - 4 \left( \frac{x}{w} - \frac{y^2}{w^2} \right) - 1 = 0
\]

Since \( w \)'s highest degree is 6 (i.e., the square of \( xy^2/w^3 \) is \( x^2y^4/w^6 \)), multiplying both sides with \( w^6 \) gives
\[
w^6 \left( \frac{x^2}{w^2} + \frac{xy^2}{w^3} - \frac{y^2}{w^2} \right)^2 - 4w^6 \left( \frac{x}{w} - \frac{y^2}{w^2} \right) - w^6 = 0
\]

Taking this \( w^6 \) into each terms yields:
\[
\left( \frac{w^3 x^2}{w^2} + \frac{w^3 xy^2}{w^3} - \frac{w^3 y^2}{w^2} \right)^2 - 4 \left( \frac{w^6 x}{w} - \frac{w^6 y^2}{w^2} \right) - w^6 = 0
\]

Therefore, the desired result is
\[
(x^2 w + xy^3 + y^2 w)^2 - 4(xw^5 - y^2 w^4) - w^6 = 0
\]
Rearranging terms gives:

\[
\left( \frac{x}{w} \right)^2 + \left( \frac{y}{w} \right)^2 + \left( \frac{z}{w} \right)^2 + 4 \left( \frac{x}{w} \right)^3 + \frac{xy}{w} + \frac{y^2z}{w^2} + 1 = 0
\]

Replacing \( x/w \) and \( y/w \) with \( x \) and \( y \), respectively, yields the desired result:

\[
(x^2 + y^2 + z)^2 + 4(x^3 + xy + y) + 1 = 0
\]

10. Find a projective transformation that maps the line determined by two points \((1,0,1)\) and \((0,1,1)\), in homogeneous coordinate, to the line at infinity.

**Solution:** A projective transformation can be represented by a \(3 \times 3\) matrix. Since both \((1,0,1)\) and \((0,1,1)\) are mapped to infinity, we have the following:

\[
\begin{bmatrix}
    a_1 \\
    b_1 \\
    0
\end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix}
    1 \\
    0 \\
    0
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
    a_2 \\
    b_2 \\
    0
\end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix}
    0 \\
    1 \\
    0
\end{bmatrix}
\]

where \(a_1, b_1, a_2, b_2,\) and \(a, \ldots, i\) are unknown. Note that in the above \((1,0,1)\) and \((0,1,1)\) are mapped to \((a_1, b_1, 0)\) and \((a_2, b_2, 0)\), which are points at infinity. Because the third component of \((a_1, b_1, 0)\) and \((a_2, b_2, 0)\) are the inner products of the third row of the transformation matrix (i.e., \([g, h, i]\)) and \((1,0,1)\) and \((0,1,1)\), respectively, we have

\[
0 = [g, h, i] \cdot \begin{bmatrix}
    1 \\
    0 \\
    1
\end{bmatrix} = g + i \quad \text{and} \quad 0 = [g, h, i] \cdot \begin{bmatrix}
    0 \\
    1 \\
    1
\end{bmatrix} = h + i
\]

From these two equation, it is clear that if \(g = h = -1\) and \(i = 1\), \((1,0,1)\) and \((0,1,1)\) are both mapped to the infinity, and, as a result, we have the following:

\[
\begin{bmatrix}
    a_1 \\
    b_1 \\
    0
\end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix}
    1 \\
    0 \\
    1
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
    a_2 \\
    b_2 \\
    0
\end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix}
    0 \\
    1 \\
    1
\end{bmatrix}
\]

Since a projective transformation matrix has to be non-singular (i.e., determinant being non-zero in this case), we can choose \(a, \ldots, f\) freely as long as the transformation is non-singular. The easiest one would be the following:

\[
\begin{bmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    -1 & -1 & 1
\end{bmatrix}
\]

The determinant of the above matrix is 1 and the transformation is non-singular. Since we have

\[
\begin{bmatrix}
    1 \\
    0 \\
    0
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix}
    1 \\
    0 \\
    1
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
    0 \\
    1 \\
    0
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix}
    1 \\
    1 \\
    1
\end{bmatrix}
\]

\((1,0,1)\) and \((0,1,1)\) are mapped to \((1,0,0)\) and \((0,1,0)\), respectively. Thus, the line determined by \((1,0,1)\) and \((0,1,1)\) is mapped to the line at infinity. \(\Box\)