CS3621 Introduction to Computing with Geometry
Solutions to Drill Exercises – Basic Concepts

1. Find the line defined by points (3,5) and (7, 14).

   Solution: The line that contains two points (a, b) and (c, d) has an equation of

   $$\frac{y-b}{x-a} = \frac{d-b}{c-a}$$

   Since $a = 3$ and $c = 7$ and are not equal, the above equation is usable. Therefore, we have

   $$\frac{y-5}{x-3} = \frac{14-5}{7-3}$$

   Rearranging terms yields the answer to this question:

   $$9x - 4y - 7 = 0$$

   You can verify this answer by plugging (3,5) and (7,14) into the equation. □

2. What is the distance from the origin to the line with equation $3x + 4y + 6 = 0$?

   Solution: From the given equation, we have $A = 3$, $B = 4$ and $C = 6$. Therefore, the distance from the origin to this line is

   $$\text{distance} = \frac{|C|}{\sqrt{A^2+B^2}} = \frac{6}{\sqrt{3^2+4^2}} = \frac{6}{5} = 1.2$$

3. Find the equation of the plane defined by base point $\mathbf{B}$ and normal vector $\mathbf{n}$. Verify your result using $\mathbf{B} = \langle 3, 4, 5\rangle$ and $\mathbf{n} = \langle 1, 1, 1\rangle$.

   Solution: The answer should be $\mathbf{X} \cdot \mathbf{n} - \mathbf{B} \cdot \mathbf{n} = 0$, where $\cdot$ is the inner product. Plugging the values for $\mathbf{B}$ and $\mathbf{n}$, we have

   $$\mathbf{X} \cdot \langle 1, 1, 1\rangle - \langle 3, 4, 5\rangle \cdot \langle 1, 1, 1\rangle = 0$$

   Expanding and rearranging yields:

   $$x + y + z - 12 = 0$$

4. What is the inner product and cross product of vectors $\mathbf{u} = \langle 1, 3, 5\rangle$ and $\mathbf{v} = \langle -2, 0, 4\rangle$. What is the length of the cross product? What is the angle between these two vectors?

   Solution: The inner and cross products are

   $$\begin{aligned}\langle 1,3,5\rangle \cdot \langle -2,0,4\rangle &= 1*(-2)+3*0+5*4 = (-2)+0+20 = 18\\ \langle 1,3,5\rangle \times \langle -2,0,4\rangle &= \left\langle \begin{vmatrix} 3 & 5 \\ 0 & 4 \end{vmatrix}, -\begin{vmatrix} 1 & 5 \\ -2 & 4 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ -2 & 0 \end{vmatrix} \right\rangle = \langle 12, -14, 6\rangle\end{aligned}$$

   Since $|\langle 1,3,5\rangle| = (1^2+3^2+5^2)^{1/2} = \sqrt{35}$ and $|\langle -2,0,4\rangle| = ((-2)^2+0^2+4^2)^{1/2} = \sqrt{20}$, the angle θ between $\langle 1,3,5\rangle$ and $\langle -2,0,4\rangle$ is computed as follows:

   $$\cos(\theta) = \frac{\langle 1,3,5\rangle \cdot \langle -2,0,4\rangle}{|\langle 1,3,5\rangle| * |\langle -2,0,4\rangle|} = \frac{18}{\sqrt{35}\sqrt{20}} \approx 0.68 \qquad \text{and} \qquad \theta = 47.13^\circ$$

   The length of $\langle 1,3,5\rangle \times \langle -2,0,4\rangle$ is computed as

   $$|\langle 1,3,5\rangle \times \langle -2,0,4\rangle| = |\langle 12,-14,6\rangle| = \sqrt{12^2+(-14)^2+6^2} = \sqrt{144+196+36} = \sqrt{376} \approx 19.39$$

5. Suppose a general second degree curve $Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$ represents a conic curve. Use the concept of *the line at infinity* to prove that the curve is an ellipse, a hyperbola, or a parabola, if and only if $B^2 - AC < 0$, $B^2 - 4AC > 0$, or $B^2 - AC = 0$. This problem looks difficult; but, it should be easy if you understand the concept of the line at infinity covered in class. Try it.

   Solution: What we covered in class is: a conic is an ellipse, hyperbola or a parabola if it intersects the line at infinity in *no* point, two points, and one point (or tangent).

   Converting the given equation to a homogeneous one yields

   $$Ax^2 + 2Bxy + Cy^2 + 2Dxw + 2Eyw + Fw^2 = 0$$

   Setting $w = 0$ means computing the intersection of the conic and the line at infinity. Thus, the intersection points are the solutions of $Ax^2+2Bxy+Cy^2 = 0$. Since all multiples of the same Euclidean point represent the same point in homogeneous coordinate, we could divide the whole equation by y^2 if $y \neq 0$; otherwise, divide the whole equation by x^2 if $y = 0$. Note that x and y cannot be both zero, because the point becomes $(0, 0, 0)$, which is not a legal point in homogeneous coordinate. Let us assume $y \neq 0$. Thus, we have

   $$A\left(\frac{x}{y}\right)^2 + 2B\left(\frac{x}{y}\right) + C = 0$$

   Now solving for x/y (rather than x and y) yields

   $$\frac{x}{y} = \frac{-B \pm \sqrt{B^2 - AC}}{A}$$

   Therefore, if $B^2 - AC < 0$ we have no real root and the conic is an ellipse; if $B^2 - AC = 0$, we have a double root and the conic is a parabola; and if $B^2 - AC > 0$, we have two real roots, and the conic is a hyperbola. □

6. Convert the following second degree equation to its equivalent matrix form:

   $$3x^2 - 5xy + 5y^2 + x + 6y + 9 = 0$$

   Solution: The answer is

   $$\begin{bmatrix} 3 & -5/2 & 1/2 \\ -5/2 & 5 & 3 \\ 1/2 & 3 & 9 \end{bmatrix}$$

7. What is the equation of the following conic curve in matrix form?

   $$\begin{bmatrix} -1 & 4 & -3 \\ 4 & 1 & 2 \\ -3 & 2 & 3 \end{bmatrix}$$

   Solution: The answer is

   $$-x^2 + 8xy + y^2 - 6x + 4y + 3 = 0$$

8. Convert the following equations to their homogeneous form:

   (a) $x^2 + y^2 + 3z^2 - 2xy + 3xz + yz + 3x - 5y - 6z + 10 = 0$
   Solution: Replacing x, y and z with x/w, y/w and z/w, respectively, we have

   $$\left(\frac{x}{w}\right)^2+\left(\frac{y}{w}\right)^2+3\left(\frac{z}{w}\right)^2-2\left(\frac{x}{w}\right)\left(\frac{y}{w}\right)+3\left(\frac{x}{w}\right)\left(\frac{z}{w}\right)+\left(\frac{y}{w}\right)\left(\frac{z}{w}\right)+3\left(\frac{x}{w}\right)-5\left(\frac{y}{w}\right)-6\left(\frac{z}{w}\right)+10 = 0$$

   Expanding the terms gives

   $$\frac{x^2}{w^2} + \frac{y^2}{w^2} + 3\frac{z^2}{w^2} - 2\frac{xy}{w^2} + 3\frac{xz}{w^2} + \frac{yz}{w^2} + 3\frac{x}{w} - 5\frac{y}{w} - 6\frac{z}{w} + 10 = 0$$

Clearing terms by multiplying w^2 to both sides yields the desired answer:

$$x^2 + y^2 + 3z^2 - 2xy + 3xz + yz + 3xw - 5yw - 6zw + 10w^2 = 0$$

(b) $(x^2 + xy^2 - y^2)^2 - 4(x - y^2) - 1 = 0$
Solution: Replacing x, y and z with x/w, y/w and z/w, respectively, we have

$$\left(\left(\frac{x}{w}\right)^2 + \frac{x}{w}\left(\frac{y}{w}\right)^2 - \left(\frac{y}{w}\right)^2\right)^2 - 4\left(\frac{x}{w} - \left(\frac{y}{w}\right)^2\right) - 1 = 0$$

This equation can be rewritten as:

$$\left(\frac{x^2}{w^2} + \frac{xy^2}{w^3} - \frac{y^2}{w^2}\right)^2 - 4\left(\frac{x}{w} - \frac{y^2}{w^2}\right) - 1 = 0$$

Since w's highest degree is 6 (*i.e.*, the square of xy^2/w^3 is x^2y^4/w^6), multiplying both sides with w^6 gives

$$w^6\left(\frac{x^2}{w^2} + \frac{xy^2}{w^3} - \frac{y^2}{w^2}\right)^2 - 4w^6\left(\frac{x}{w} - \frac{y^2}{w^2}\right) - w^6 = 0$$

Taking this w^6 into each terms yields:

$$\left(w^3\frac{x^2}{w^2} + w^3\frac{xy^2}{w^3} - w^3\frac{y^2}{w^2}\right)^2 - 4\left(w^6\frac{x}{w} - w^6\frac{y^2}{w^2}\right) - w^6 = 0$$

Therefore, the desired result is

$$\left(x^2w + xy^3 + y^2w\right)^2 - 4\left(xw^5 - y^2w^4\right) - w^6 = 0$$

9. Convert the following homogeneous equations to their non-homogeneous form:

(a) $xy^4 + x^2y^2w + y^4w + 2xyw^3 - 3xw^4 + w^5 = 0$
Solution: The degree of this homogeneous equation is 5. Thus, dividing the whole equation by w^5 gives

$$\frac{xy^4}{w^5} + \frac{x^2y^2w}{w^5} + \frac{y^4w}{w^5} + 2\frac{xyw^3}{w^5} - 3\frac{xw^4}{w^5} + \frac{w^5}{w^5} = 0$$

Rearrange the terms:

$$\frac{x}{w}\frac{y^4}{w^4} + \frac{x^2}{w^2}\frac{y^2}{w^2}\frac{w}{w} + \frac{y^4}{w^4}\frac{w}{w} + 2\frac{x}{w}\frac{y}{w}\frac{w^3}{w^3} - 3\frac{x}{w}\frac{w^4}{w^4} + 1 = 0$$

Replacing x/w, y/w and z/w with x, y and z, respectively, yields the desired result:

$$xy^4 + x^2y^2 + y^4 + 2xy - 3x + 1 = 0$$

(b) $(x^2 + y^2 + zw)^2 + 4w(x^3 + xyw + yw^2) + w^4 = 0$
Solution: The degree of this equation is 4. Dividing the whole equation with w^4 gives:

$$\frac{(x^2 + y^2 + zw)^2}{w^4} + 4\frac{w(x^3 + xyw + yw^2)}{w^4} + \frac{w^4}{w^4} = 0$$

Moving w^4 into each term gives:

$$\left(\frac{x^2}{w^2} + \frac{y^2}{w^2} + \frac{zw}{w^2}\right)^2 + 4\frac{w}{w}\left(\frac{x^3}{w^3} + \frac{xyw}{w^3} + \frac{yw^2}{w^3}\right) + 1 = 0$$

Rearranging terms gives:

$$\left(\left(\frac{x}{w}\right)^2+\left(\frac{y}{w}\right)^2+\frac{z}{w}\frac{w}{w}\right)^2+4\left(\left(\frac{x}{w}\right)^3+\frac{x}{w}\frac{y}{w}\frac{w}{w}+\frac{y}{w}\frac{w^2}{w^2}\right)+1=0$$

Replacing x/w and y/w with x and y, respectively, yields the desired result:

$$\left(x^2+y^2+z\right)^2+4\left(x^3+xy+y\right)+1=0$$

10. Find a projective transformation that maps the line determined by two points $(1,0,1)$ and $(0,1,1)$, in homogeneous coordinate, to the line at infinity.

Solution: A projective transformation can be represented by a 3×3 matrix. Since both $(1,0,1)$ and $(0,1,1)$ are mapped to infinity, we have the following:

$$\begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_2 \\ b_2 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

where a_1, b_1, a_2, b_2, and a, ... i are unknown. Note that in the above $(1,0,1)$ and $(0,1,1)$ are mapped to $(a_1,b_1,0)$ and $(a_2,b_2,0)$, which are points at infinity. Because the third component of $(a_1,b_1,0)$ and $(a_2,b_2,0)$ are the inner products of the third row of the transformation matrix (*i.e.*, $[g,h,i]$) and $(1,0,1)$ and $(0,1,1)$, respectively, we have

$$0=[g,h,i]\cdot\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}=g+i \quad \text{and} \quad 0=[g,h,i]\cdot\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}=h+i$$

From these two equation, it is clear that if $g=h=-1$ and $i=1$, $(1,0,1)$ and $(0,1,1)$ are both mapped to the infinity, and, as a result, we have the following:

$$\begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_2 \\ b_2 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Since a projective transformation matrix has to be non-singular (*i.e.*, determinant being non-zero in this case), we can choose a, ..., f freely as long as the transformation is non-singular. The easiest one would be the following:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

The determinant of the above matrix is 1 and the transformation is non-singular. Since we have

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$(1,0,1)$ and $(0,1,1)$ are mapped to $(1,0,0)$ and $0,1,0)$, respectively. Thus, the line determined by $(1,0,1)$ and $(0,1,1)$ is mapped to the line at infinity. $\square$